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Superevolution

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Abstract

Usually, in supersymmetric theories, it is assumed that the time evolution of states is determined by the Hamiltonian, through the Schrödinger equation. Here we explore the superevolution of states in superspace, in which the supercharges are the principal operators. The superevolution equation is consistent with the Schrödinger equation, but it avoids the usual degeneracy between bosonic and fermionic states. We discuss superevolution in supersymmetric quantum mechanics and in a simple supersymmetric field theory.

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1. Introduction

Supersymmetry is a beautiful symmetry, extending the Poincaré symmetry of space–time. The supersymmetry generators are spinorial in character, and they relate bosonic states with integer spin to fermionic states of half-integer spin. Supersymmetry is present in a wide range of theoretical settings—quantum field theory, supergravity and superstring theory. Yet there is no evidence so far of the physical realization of supersymmetry. Even if supersymmetry were established through the discovery of heavier partner particles of the elementary particles we know, it would still not be exact supersymmetry but only an approximate, broken form.

A simpler version of supersymmetry occurs in supersymmetric quantum mechanics (see [1] for a recent discussion of the various formulations). Here, the states that are related by supersymmetry may or may not differ in spin. Some rather special systems with exact or approximate supersymmetry, in this more limited sense, are physically realized. Certain nuclei have a sequence of excited states labelled by increasing energy and angular momentum. The energy levels of one nucleus with half-integer angular momenta have approximately the same spacing as the combined energy levels of two close-by nuclei with integer angular momenta. These states can be modelled using a supersymmetry algebra [2]. Another example is an electron in a magnetic field, restricted to a two-dimensional plane [3, 4]. The magnetic field need not be uniform. Here, the states of the electron with spin up are paired by a supersymmetry operator to states of the same energy with spin down. Only the zero-energy states are unpaired. (In this, and some other quantum mechanical examples, the states related by supersymmetry are not really distinguished by the dichotomy 'bosonic/fermionic', but it is convenient to use this terminology, and we will do so in what follows.) In both these examples, the states related by supersymmetry are physically distinct, but have the same energy.

Yet there are many physical systems where there is a hint of supersymmetry, without the accompanying energy degeneracy of states. In particular, part of a supersymmetric system can be something well known, and physically realized. A nice example is the supersymmetric quantum mechanical Coulomb system. This has been explored in an arbitrary dimension by Kirchberg *et al* [5], but let us just consider the version in ordinary three-dimensional space. The superpotential which gives the Coulomb potential is quite simple, being $\frac{1}{2}\alpha r$, with r the distance to the source and α a measure of its strength. In the formalism of Witten [6], applied to this example, the wavefunctions are differential forms in $\mathbb{R}^3$, and the Hamiltonian commutes with the degree of the form. So the Hilbert space splits up into the subspaces of 0-forms, 1-forms, 2-forms and 3-forms. Acting on 0-forms, the Hamiltonian is $-\nabla^2 + \frac{\alpha}{r} + \frac{\alpha^2}{4}$, which for $\alpha < 0$ is essentially the Hamiltonian of the standard hydrogen atom, but with energy levels shifted so that the ground-state energy is zero. It is tempting to say that this latter Hamiltonian is supersymmetric because it is part of a larger, truly supersymmetric system. The full system has further Hamiltonians, acting on 1-forms, 2-forms and 3-forms. These are physically meaningful (for example, the Hamiltonian on 3-forms is the repulsive Coulomb Hamiltonian), but they do not occur simultaneously with the standard Coulomb Hamiltonian in the hydrogen atom. Another example of this type is the Planck oscillator. This is the standard quantum harmonic oscillator but with its ground-state energy shifted to zero. The quantized electromagnetic field can be regarded as an infinite set of such oscillators, labelled by momentum and polarization. A photon is the first excited state of one of these oscillators. The Planck oscillator also occurs naturally in a supersymmetric context, but is then accompanied by a second, fermionic oscillator which has no physical role in the theory of electromagnetic radiation.

Our goal, in this paper, is to find a convincing reinterpretation of supersymmetric systems, which avoids the degeneracy between bosonic and fermionic states. Our hope is to obtain a new understanding of physical systems previously regarded as, say, the bosonic part of a supersymmetric system. We are particularly interested in quantum field theory examples, where we want to retain the advantages of supersymmetry, but avoid the mass degeneracy of bosonic and fermionic particles. However, before studying quantum field theory, we shall explore some quantum mechanical models.

Let us recall the relationship between factorizable Hamiltonians [7–9] and the structure of supersymmetric quantum mechanics in one space dimension [10–12]. Suppose the Hamiltonian H_0 of a quantum particle can be factorized as

$$H_0 = A^\dagger A. \tag{1.1}$$

Then, there is a related Hamiltonian, namely $H_1 = AA^\dagger$. H_0 and H_1 are called partner Hamiltonians. The standard example is where $A = \frac{1}{\sqrt{2}}\big(\frac{\mathrm{d}}{\mathrm{d}x} + W(x)\big)$ and $A^\dagger = \frac{1}{\sqrt{2}}\big(-\frac{\mathrm{d}}{\mathrm{d}x} + W(x)\big)$, so

$$2H_0 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + W(x)^2 - \frac{\mathrm{d}W(x)}{\mathrm{d}x}, \qquad 2H_1 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + W(x)^2 + \frac{\mathrm{d}W(x)}{\mathrm{d}x}. \tag{1.2}$$

H_0 and H_1 both have non-negative spectrum, and any positive eigenvalue of H_0 is also a positive eigenvalue of H_1. This is because if $A^\dagger A\psi = E\psi$, then $AA^\dagger A\psi = EA\psi$. So if ψ is an eigenfunction of H_0 with eigenvalue E, then $A\psi$ is an eigenfunction of H_1 with eigenvalue E. This argument breaks down if $A\psi = 0$, but in that case $E = 0$.

H_0 could easily be a physical Hamiltonian for a particle in one dimension, and we learn that there is a related physical Hamiltonian H_1 with almost the same spectrum. However, the system does not simultaneously have both Hamiltonians. The Planck oscillator example is $W(x) = x$, where

$$2H_0 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2 - 1, \qquad 2H_1 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2 + 1. \tag{1.3}$$

H_0 has spectrum $0, 1, 2, \ldots$ and H_1 has spectrum $1, 2, \ldots$.

In the supersymmetric quantum mechanical analogue of partner Hamiltonians, the Hilbert space is of the form $\mathcal{H} = \mathcal{H}_\mathrm{b} \oplus \mathcal{H}_\mathrm{f}$, and the wavefunction is a pair of ordinary functions $\binom{\psi_0}{\psi_1}$, where ψ_0 is interpreted as the bosonic part and ψ_1 as the fermionic part. The Hamiltonian has the diagonal form

$$H = \begin{pmatrix} H_0 & 0 \\ 0 & H_1 \end{pmatrix} = \begin{pmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{pmatrix}. \tag{1.4}$$

There is a Hermitian supersymmetry operator

$$Q = \begin{pmatrix} 0 & A^\dagger \\ A & 0 \end{pmatrix} \tag{1.5}$$

and the grading (Witten) operator

$$K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.6}$$

The complete supersymmetry algebra is

$$K^2 = 1, \qquad \{K, Q\} = 0, \qquad Q^2 = H, \tag{1.7}$$

from which it follows that Q commutes with H. This system has the two Hamiltonians H_0 and H_1 acting on the different sectors, and it has degenerate fermionic and bosonic states (with $E \neq 0$), connected by the action of Q.

So far we have mentioned only Hamiltonians and their spectra, but quantum mechanics is about the time evolution of states via the Schrödinger equation, so let us look at this. In supersymmetric quantum mechanics, the usual Schrödinger equation is

$$\begin{pmatrix} \mathrm{i}\frac{\partial\psi_0}{\partial t} \\ \mathrm{i}\frac{\partial\psi_1}{\partial t} \end{pmatrix} = \begin{pmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix}, \tag{1.8}$$

so ψ_0 and ψ_1 evolve according to their respective Hamiltonians. A key point is that ψ_0 and ψ_1 at some initial time, say $t = 0$, are independent. This leads to the degeneracy between fermionic and bosonic states. Q maps bosonic into fermionic states and vice versa, but plays no direct role in the time evolution.

One could impose a superselection rule forbidding linear superpositions of fermionic and bosonic states. Even so, the initial state could be either fermionic or bosonic, and there would still be two independent states with the same energy.

An idea we have considered, but which is not our final proposal, is to restrict further, and impose the condition that $\psi_1 = 0$ at the initial time. ψ_1 would then be zero for all time. In this way, one would recover the Schrödinger equation for one of the partner Hamiltonians, and there would be no degeneracy. The ordinary quantum mechanics with Hamiltonian H_0 and wavefunction ψ_0 becomes, in this way, part of a larger supersymmetric structure. One could simply pronounce that this restricted Schrödinger equation is supersymmetric. This idea does not properly exploit the supersymmetry, because there is no explicit role for the supercharge Q.

However, there is another route which captures the supersymmetric spirit of the problem, leading to a very similar outcome. This involves an evolution equation in a superextension of physical time, and ψ_1 is non-vanishing. We describe this next.

2. Superevolution in supersymmetric quantum mechanics

It has been known for a long time [13] that the Schrödinger equation of supersymmetric quantum mechanics has a 'square root' in which the evolution is determined by the supercharge Q. Let the operators be as before. The superevolution equation is

$$\begin{pmatrix} \mathrm{i}\frac{\partial\psi_0}{\partial t} \\ \psi_1 \end{pmatrix} = \begin{pmatrix} 0 & A^\dagger \\ A & 0 \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix}, \tag{2.1}$$

which can be written more compactly as

$$\begin{pmatrix} \mathrm{i}\frac{\partial}{\partial t} & -A^\dagger \\ -A & 1 \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} = 0. \tag{2.2}$$

The equations for the components are

$$\mathrm{i}\frac{\partial\psi_0}{\partial t} = A^\dagger\psi_1, \tag{2.3}$$

$$\psi_1 = A\psi_0. \tag{2.4}$$

Substituting (2.4) into (2.3) we see that ψ_0 obeys its Schrödinger equation, $\mathrm{i}\frac{\partial\psi_0}{\partial t} = A^\dagger A\psi_0$. Taking the time derivative of (2.4) and substituting (2.3) we see that ψ_1 obeys its Schrödinger equation $\mathrm{i}\frac{\partial\psi_1}{\partial t} = AA^\dagger\psi_1$. However, ψ_0 and ψ_1 are not independent. Indeed, the general solution of the superevolution equation is obtained by taking an arbitrary solution of the Schrödinger equation for ψ_0, and then setting $\psi_1 = A\psi_0$. We may regard ψ_1 as a shadow of the physical state ψ_0.

In some ways, postulating the superevolution equation is hardly different from the earlier idea of just taking ψ_0 evolving with its corresponding Hamiltonian. But it has a more supersymmetric flavour and still achieves the desired result of avoiding the degeneracy of fermionic and bosonic states. For each energy eigenstate of H_0 there is just one solution of (2.1), up to an overall normalization constant.

Apart from the lack of independence of ψ_0 and ψ_1, there is another significant difference between the superevolution equation and the separate Schrödinger equations for ψ_0 and ψ_1. This concerns the zero-energy states. First, suppose that H_0 has an eigenfunction ϕ with zero eigenvalue. Then, $0 = \langle\phi|H_0|\phi\rangle = \langle\phi|A^\dagger A|\phi\rangle = \langle A\phi|A\phi\rangle$, so $A\phi = 0$. Therefore, the corresponding solution of (2.1) is $\psi_0 = \phi, \psi_1 = 0$. Consistent with zero energy, there is no time dependence. Second, suppose that H_1 has an eigenfunction $\widetilde{\phi}$ with zero eigenvalue. Then, $A^\dagger\widetilde{\phi} = 0$. Equation (2.3) is solved by setting $\psi_1 = \widetilde{\phi}$ and ψ_0 to be any time-independent function, but for (2.4) to be satisfied one requires that $\widetilde{\phi} = A\psi_0$. This cannot be solved, since it implies $\langle\widetilde{\phi}|\widetilde{\phi}\rangle = \langle A\psi_0|\widetilde{\phi}\rangle = \langle\psi_0|A^\dagger\widetilde{\phi}\rangle = 0$, and hence $\widetilde{\phi} = 0$, a contradiction. We conclude that zero-energy states of H_0 have corresponding solutions of the superevolution equation, but zero-energy states of H_1 do not.

There is a supertime formulation of the superevolution equation. Let us extend the time line to a supertime $\mathbb{R}^{1|1}$ with coordinates (t, τ). τ commutes with t but is odd, and $\tau^2 = 0$. The supertime evolution operator D is defined to be

$$D = \frac{\partial}{\partial\tau} - \tau\mathrm{i}\frac{\partial}{\partial t}. \tag{2.5}$$

This acts on a wavefunction Ψ which is a function of t, τ and x. Ψ has the expansion

$$\Psi = \Psi_0 + \tau\Psi_1, \tag{2.6}$$

where $\Psi_0 \in \mathcal{H}_\mathrm{b}$ and $\Psi_1 \in \mathcal{H}_\mathrm{f}$ depend only on t and x. Therefore,

$$D\Psi = \left(\frac{\partial}{\partial\tau} - \tau\mathrm{i}\frac{\partial}{\partial t}\right)(\Psi_0 + \tau\Psi_1) = \Psi_1 - \tau\mathrm{i}\frac{\partial\Psi_0}{\partial t}. \tag{2.7}$$

Let us now extend the earlier definition of Q, by linearity, to Ψ, with Q acting on Ψ_0 and Ψ_1 as Q previously acted on ψ_0 and ψ_1. Q and τ are assumed to anticommute. Therefore,

$$Q\Psi = Q\Psi_0 - \tau Q\Psi_1 = A\Psi_0 - \tau A^\dagger\Psi_1, \tag{2.8}$$

with $A\Psi_0 \in \mathcal{H}_\mathrm{f}$ and $A^\dagger\Psi_1 \in \mathcal{H}_\mathrm{b}$. The superevolution equation is taken as

$$D\Psi = Q\Psi. \tag{2.9}$$

Combining (2.7) and (2.8), we see that in components

$$\mathrm{i}\frac{\partial\Psi_0}{\partial t} = A^\dagger\Psi_1, \tag{2.10}$$

$$\Psi_1 = A\Psi_0. \tag{2.11}$$

It is easy to verify, abstractly or by acting on Ψ, that

$$D^2 = -\mathrm{i}\frac{\partial}{\partial t}, \tag{2.12}$$

so D is the square root of (minus) the time evolution operator that occurs in the Schrödinger equation. One can check directly by acting on Ψ that $DQ + QD = 0$. The superevolution equation is therefore a consistent square root of the Schrödinger equation, because it implies that

$$-\mathrm{i}\frac{\partial\Psi}{\partial t} = D^2\Psi = DQ\Psi = -QD\Psi = -Q^2\Psi = -H\Psi. \tag{2.13}$$

We have made a notational distinction between ψ_0, ψ_1, which are ordinary functions of x and t, and Ψ_0, Ψ_1, for the following reason. It is best to regard Ψ_0 as even and Ψ_1 as odd. This can be made explicit by extending space, with coordinate x, to a superspace $\mathbb{R}^{1|1}$ with coordinates (x, θ), where $\theta^2 = 0$ and θ and τ anticommute. Then, let $\Psi_0 = \psi_0$ and $\Psi_1 = \theta\psi_1$. The total wavefunction Ψ becomes the even expression

$$\Psi = \psi_0 + \tau\theta\psi_1. \tag{2.14}$$

The operator D is as before, but Q becomes

$$Q = \theta A + \frac{\partial}{\partial\theta}A^\dagger. \tag{2.15}$$

Note that both $D\Psi$ and $Q\Psi$ are odd. The superevolution equation (2.9) takes the form

$$\left(\frac{\partial}{\partial\tau} - \tau\mathrm{i}\frac{\partial}{\partial t}\right)(\psi_0 + \tau\theta\psi_1) = \left(\theta A + \frac{\partial}{\partial\theta}A^\dagger\right)(\psi_0 + \tau\theta\psi_1). \tag{2.16}$$

This simplifies to

$$\theta\psi_1 - \tau\mathrm{i}\frac{\partial\psi_0}{\partial t} = \theta A\psi_0 - \tau A^\dagger\psi_1. \tag{2.17}$$

Comparing coefficients of θ and τ, we recover the component equations

$$\mathrm{i}\frac{\partial\psi_0}{\partial t} = A^\dagger\psi_1, \tag{2.18}$$

$$\psi_1 = A\psi_0, \tag{2.19}$$

as before.

The superevolution equation (2.9) was considered by Friedan and Windey [13] in the context of a spinning supersymmetric particle, whose supersymmetry charge Q is the Dirac operator, and using it they gave a novel proof of the Atiyah–Singer index theorem. The initial state at $t = \tau = 0$ was taken to be a spatial delta function, with vanishing dependence on odd spatial coordinates like θ. In contrast, we would allow the initial state ψ_0 $(t = 0)$ to be an arbitrary, ordinary function of x. More importantly, our proposal is for the physical realization of supersymmetric quantum mechanics, and not just a mathematical application.

Superevolution equations have also been considered by Rogers [14], who constructed a path integral representation for the finite (Euclidean) supertime evolution operator $\mathrm{e}^{-Ht-Q\tau}$. The initial state at $t = 0$, $\tau = 0$ (in the context of one-dimensional supersymmetric quantum mechanics) was taken to be a general function of x and θ, $\Psi = \phi_0(x) + \theta\phi_1(x)$; however, we would impose the restriction that Ψ is even. The wavefunction at $\tau = 0$ is then purely bosonic, and the fermionic part only occurs multiplied by τ. This avoids the degeneracy between bosonic and fermionic states. The physical wavefunction at a later time can be identified with $\Psi(t, \tau = 0)$.

In quantized supergravity, the fundamental role of the supercharges has also been stressed. The standard constraint of quantum gravity, that the Hamiltonian annihilates physical states, can be replaced by the constraint that the supercharges annihilate physical states [15–17].

Returning now to quantum mechanics, recall that not just the evolution of the wavefunction is important. One must also consider observables and their expectation values. In superevolution, we are regarding ψ_0 as the physical wavefunction and $\psi_1 = A\psi_0$ as its shadow. We propose that an observable should be a Hermitian operator acting on ψ_0. Let us define a normalized wavefunction to be one satisfying $\langle\psi_0|\psi_0\rangle = 1$. For such a wavefunction, the expectation value of an observable O is $\langle\psi_0|O|\psi_0\rangle$.

If ψ_0 is normalized, then the shadow wavefunction satisfies the normalization $\langle\psi_1|\psi_1\rangle = \langle A\psi_0|A\psi_0\rangle = \langle\psi_0|H_0|\psi_0\rangle = E$, where E is the energy expectation value. There are also shadow observables $\widetilde{O}$ acting on ψ_1, but these can be related to standard observables. The observable O related to $\widetilde{O}$ is given by

$$\langle\psi_0|O|\psi_0\rangle = \langle\psi_1|\widetilde{O}|\psi_1\rangle, \tag{2.20}$$

so $O = A^\dagger\widetilde{O}A$. The expectation value of $\widetilde{O}$ is defined as

$$\frac{\langle\psi_1|\widetilde{O}|\psi_1\rangle}{\langle\psi_1|\psi_1\rangle} = \frac{1}{E}\langle\psi_0|O|\psi_0\rangle. \tag{2.21}$$

For example, if $\widetilde{O} = 1$, then $O = H_0$ and the expectation value of $\widetilde{O}$ is 1. If $\widetilde{O} = H_1 = AA^\dagger$, then $O = (H_0)^2$ and the expectation value is E. The shadow observables do not make sense in a state with $E = 0$, because ψ_1 then vanishes.

3. Supersymmetry and differential forms

Witten [6] has formulated a large class of supersymmetric quantum mechanical models in which the wavefunction is a differential form on some finite-dimensional Riemannian manifold M. Many examples of supersymmetric quantum mechanics, including those discussed in sections 1 and 2, are special cases.

The basic model just involves the geometry of M. Let M have dimension n and let $\Omega^{\mathrm{ev}}(\Omega^{\mathrm{odd}})$ denote the space of forms of even (odd) degree. The complete Hilbert space is $\mathcal{H} = \mathcal{H}_{\mathrm{b}}\oplus\mathcal{H}_{\mathrm{f}}$ where $\mathcal{H}_{\mathrm{b}} = \Omega^{\mathrm{ev}}$ and $\mathcal{H}_{\mathrm{f}} = \Omega^{\mathrm{odd}}$. A wavefunction is therefore a pair $\Psi = \binom{\omega^{\mathrm{ev}}}{\omega^{\mathrm{odd}}}$, where $\omega^{\mathrm{ev}} \in \Omega^{\mathrm{ev}}$ is regarded as bosonic and $\omega^{\mathrm{odd}} \in \Omega^{\mathrm{odd}}$ as fermionic.

The supersymmetry operator Q is constructed from the de Rham exterior derivative d and its adjoint δ. ($\delta = *d*$ acting on Ω^{ev} when n is odd and $\delta = -*d*$ otherwise. $*$ is the Hodge duality operator, whose definition requires a Riemannian metric.) d increases the degree of a form by 1 and δ decreases the degree by 1, so both operators map even forms to odd forms and vice versa. d and δ have the properties $d^2 = 0$ and $\delta^2 = 0$, so $(d+\delta)^2 = d\delta + \delta d$, the Laplace–Beltrami operator acting on forms on M. The supersymmetry operators are

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & d+\delta \\ d+\delta & 0 \end{pmatrix}, \tag{3.1}$$

$$H = \frac{1}{2} \begin{pmatrix} d\delta + \delta d & 0 \\ 0 & d\delta + \delta d \end{pmatrix}, \tag{3.2}$$

with K as before. These satisfy the algebra (1.7). Note that, formally, Q and H act in the same way on Ω^{ev} and Ω^{odd}, but this is rather an illusion, since the detailed formulae depend on the degree.

A key observation of Witten is that the operators above can be modified to include a real-valued, superpotential function h defined on M. One simply replaces d by $d_h = \mathrm{e}^{-h} d \mathrm{e}^{h}$ and δ by $\delta_h = \mathrm{e}^{h} \delta \mathrm{e}^{-h}$ in Q and H. (Note that this is not a trivial conjugation, as it would be if δ_h were $\mathrm{e}^{-h} \delta \mathrm{e}^{h}$.) The algebra (1.7) is still satisfied.

The usual Schrödinger evolution of states, governed by the Hamiltonian H, can be replaced by a superevolution, with component equations

$$\mathrm{i} \frac{\partial}{\partial t} \omega^{\mathrm{ev}} = \frac{1}{\sqrt{2}} (d_h + \delta_h) \omega^{\mathrm{odd}}, \tag{3.3}$$

$$\omega^{\mathrm{odd}} = \frac{1}{\sqrt{2}} (d_h + \delta_h) \omega^{\mathrm{ev}}. \tag{3.4}$$

This again avoids the degeneracy between bosonic and fermionic states connected by Q, that occurs with the Schrödinger evolution. A special case reproduces the one-dimensional supersymmetric quantum mechanical model of section 2. Choose $M = \mathbb{R}$ and set $W(x) = \frac{\mathrm{d}h(x)}{\mathrm{d}x}$. The wavefunction is the pair

$$\Psi = \begin{pmatrix} \psi_0 \\ \psi_1 \mathrm{d}x \end{pmatrix}, \tag{3.5}$$

where ψ_0 and ψ_1 are ordinary functions of x and t. The abstract θ of section 2 is here replaced by $\mathrm{d}x$. Then, (3.3) and (3.4) reduce to

$$\begin{aligned} \mathrm{i} \frac{\partial}{\partial t} \psi_0 &= \frac{1}{\sqrt{2}} (d_h + \delta_h) \psi_1 \, \mathrm{d}x \\ &= -\frac{1}{\sqrt{2}} (\mathrm{e}^{h} * d * (\mathrm{e}^{-h} \psi_1 \, \mathrm{d}x)) \\ &= \frac{1}{\sqrt{2}} \left(-\frac{\partial \psi_1}{\partial x} + \frac{\mathrm{d}h}{\mathrm{d}x} \psi_1 \right) \\ &= A^{\dagger} \psi_1 \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \psi_1 \, \mathrm{d}x &= \frac{1}{\sqrt{2}} (d_h + \delta_h) \psi_0 \\ &= \frac{1}{\sqrt{2}} \mathrm{e}^{-h} d(\mathrm{e}^{h} \psi_0) \\ &= \frac{1}{\sqrt{2}} \left(\frac{\partial \psi_0}{\partial x} + \frac{\mathrm{d}h}{\mathrm{d}x} \psi_0 \right) \mathrm{d}x \\ &= (A \psi_0) \, \mathrm{d}x, \end{aligned} \tag{3.7}$$

which reproduce equations (2.3) and (2.4).

The supertime version of the superevolution equations (3.3) and (3.4) can be expressed entirely in terms of differential forms, and standard operations on forms. Recall that the fermionic wavefunction ω^{odd} is odd as it stands. We identify τ with the 1-form $\mathrm{d}t$, and then extend Witten's formalism by making the wavefunction into a differential form on $\widetilde{M} = M \times (-\infty, \infty)$, where the second factor is the time axis. The wavefunction is now taken to be

$$\widetilde{\Psi} = \omega^{\mathrm{ev}} + \mathrm{d}t \wedge \omega^{\mathrm{odd}}, \tag{3.8}$$

an even form on $\widetilde{M}$. The operator $D = \frac{\partial}{\partial \tau} - \tau \mathrm{i} \frac{\partial}{\partial t}$ becomes

$$\widetilde{D} = \iota_{\frac{\partial}{\partial t}} - \mathrm{d}t\, \mathrm{i} \frac{\partial}{\partial t}, \tag{3.9}$$

where $\iota_{\frac{\partial}{\partial t}}$ is the inner product operator that cancels a $\mathrm{d}t$ factor immediately to the right (and gives zero acting on a form with no $\mathrm{d}t$ in it), and $\mathrm{d}t$ acts by left exterior multiplication. Q is replaced by the essentially identical $\widetilde{Q} = \frac{1}{\sqrt{2}}(d_h + \delta_h)$, which is defined to act just on the M variables, and which anticommutes with $\mathrm{d}t$. In this extended formalism, the superevolution equation becomes

$$\widetilde{D}\widetilde{\Psi} = \widetilde{Q}\widetilde{\Psi}. \tag{3.10}$$

This reduces to the component equations (3.3) and (3.4).

Note that the Witten model is not relativistic. Even if $M = \mathbb{R}^n$ and there is no superpotential, the superevolution equation is not Lorentz invariant in $\mathbb{R}^{n+1}$.

4. Superevolution in field theory

In this section we consider the simplest supersymmetric quantum field theory in 1+1 dimensions, the theory of one real scalar field and one Majorana fermion field [18], and present its superevolution equations. But first, we present the conventional interpretation of the field theory and its Schrödinger equations.

It is standard to write down the Lagrangian first, and then canonically quantize. However, the Majorana condition implies that the Majorana field is conjugate to itself, and this leads to some ambiguities in factors of 2. This difficulty can be resolved using a Dirac constraint formalism but we will not go through this. Instead, we shall simply state the canonical commutation and anticommutation relations for the field operators, and give the algebra of supersymmetry operators.

Let x (or y) denote the spatial coordinate. The scalar field $\phi(x)$ and its conjugate momentum $\pi(x)$ are independent Hermitian operators at each point. In the Schrödinger representation, they have no time dependence.

The Dirac matrices obey $(\gamma^0)^2 = 1$, $(\gamma^1)^2 = -1$ and $\gamma^0\gamma^1 + \gamma^1\gamma^0 = 0$. We shall use the Majorana representation for these:

$$\gamma^0 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}. \tag{4.1}$$

The Majorana spinor field $\psi(x) = \binom{\psi_1(x)}{\psi_2(x)}$ has two components, both of which are Hermitian operators.

The non-vanishing canonical commutation and anticommutation relations are

$$[\phi(x), \pi(y)] = \mathrm{i}\delta(x - y), \tag{4.2}$$

$$\{\psi_\alpha(x), \psi_\beta(y)\} = \delta_{\alpha\beta}\delta(x - y), \tag{4.3}$$

with all commutators $[\phi, \phi]$, $[\pi, \pi]$ and $[\phi, \psi_\alpha]$, $[\pi, \psi_\alpha]$ vanishing.

The Hamiltonian is

$$H = \frac{1}{2}\int \left(\pi^2 + (\partial_x\phi)^2 + W(\phi)^2 + \mathrm{i}\psi_1\partial_x\psi_1 - \mathrm{i}\psi_2\partial_x\psi_2 + 2\mathrm{i}\frac{\mathrm{d}W(\phi)}{\mathrm{d}\phi}\psi_1\psi_2\right)\mathrm{d}x, \tag{4.4}$$

where $W(\phi)$ is an arbitrary function of ϕ, usually assumed to be a polynomial. The total momentum operator is[1]

$$P = \frac{1}{2}\int (2\pi\,\partial_x\phi + \mathrm{i}\psi_1\partial_x\psi_1 + \mathrm{i}\psi_2\partial_x\psi_2)\,\mathrm{d}x. \tag{4.5}$$

It is best to combine these into the combinations

$$H + P = \frac{1}{2}\int \left((\pi + \partial_x\phi)^2 + W(\phi)^2 + 2\mathrm{i}\psi_1\partial_x\psi_1 + 2\mathrm{i}\frac{\mathrm{d}W(\phi)}{\mathrm{d}\phi}\psi_1\psi_2\right)\mathrm{d}x, \tag{4.6}$$

$$H - P = \frac{1}{2}\int \left((\pi - \partial_x\phi)^2 + W(\phi)^2 - 2\mathrm{i}\psi_2\partial_x\psi_2 + 2\mathrm{i}\frac{\mathrm{d}W(\phi)}{\mathrm{d}\phi}\psi_1\psi_2\right)\mathrm{d}x. \tag{4.7}$$

The two supercharges are

$$Q_1 = \int ((\pi + \partial_x\phi)\psi_1 - W(\phi)\psi_2)\,\mathrm{d}x, \tag{4.8}$$

$$Q_2 = \int ((\pi - \partial_x\phi)\psi_2 + W(\phi)\psi_1)\,\mathrm{d}x. \tag{4.9}$$

The theory simplifies to a free theory if $W(\phi) = m\phi$. The particles associated with the quantized scalar and Majorana field then both have mass m.

After a somewhat long calculation, using the canonical (anti)commutation relations, one can verify that the above operators obey the supersymmetry algebra

$$Q_1^2 = H + P, \tag{4.10}$$

$$Q_2^2 = H - P, \tag{4.11}$$

$$Q_1Q_2 + Q_2Q_1 = 0. \tag{4.12}$$

These imply that Q_1 and Q_2 commute with both H and P. Formally, a boundary contribution appears as a central charge on the right-hand side of (4.12) but it vanishes if we suppose that ϕ has equal vacuum expectation values as $x \to \pm\infty$.

As an example of part of the calculation of Q_1^2, consider the square of the term involving $\pi\psi_1$. Symmetrizing in the spatial variables of integration x and y, this becomes

$$\frac{1}{2}\iint (\pi(x)\psi_1(x)\pi(y)\psi_1(y) + \pi(y)\psi_1(y)\pi(x)\psi_1(x))\,\mathrm{d}x\,\mathrm{d}y, \tag{4.13}$$

and since π commutes with itself and with ψ_1 this simplifies to

$$\begin{aligned}
&\frac{1}{2}\iint \pi(x)\pi(y)(\psi_1(x)\psi_1(y) + \psi_1(y)\psi_1(x))\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad = \frac{1}{2}\int \pi(x)\pi(y)\delta(x-y)\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad = \frac{1}{2}\int (\pi(x))^2\,\mathrm{d}x.
\end{aligned} \tag{4.14}$$

In the Schrödinger picture, states evolve in time according to the Hamiltonian. Let Ψ denote the complete quantum state and T the time. Ψ obeys the Schrödinger equation

$$\mathrm{i}\frac{\partial\Psi}{\partial T} = H\Psi. \tag{4.15}$$

[1] Our signs are such that H and P are the time and space components of the covariant 2-vector P_μ.

If Ψ is an eigenstate of H with total energy E, then

$$\Psi(T) = \Psi(0)\,\mathrm{e}^{-\mathrm{i}ET}. \tag{4.16}$$

One normally regards $\Psi(0)$ as a superposition of states completely specified by the numbers and momenta of the various particles in the theory. For example, in the free theory, for one scalar particle of mass m and momentum p, the energy is $E_p = \sqrt{p^2+m^2}$, and one would write $\Psi(0) = |p\rangle$ and $\Psi(T) = |p\rangle\,\mathrm{e}^{-\mathrm{i}E_pT}$.

However, for our purposes this is inadequate, because it does not give a satisfactory representation of the total momentum and spatial displacement operators. We need to regard states as functions of time T and of the spatial centre of mass coordinate X. A general state is written as $\Psi(T, X)$. X is well defined for field configurations that are localized in space and approach the classical vacuum at infinity, but for the plane-wave field configurations that occur in free field theory, a conventional choice must be made (see below).

We can now impose more symmetrical time and space evolution equations on the state Ψ, namely

$$\mathrm{i}\frac{\partial\Psi}{\partial T} = H\Psi, \tag{4.17}$$

$$\mathrm{i}\frac{\partial\Psi}{\partial X} = P\Psi. \tag{4.18}$$

If, as usual, Ψ is an eigenstate of both H and P, with the eigenvalues E and P' being the total energy and momentum, then

$$\Psi(T, X) = \Psi(0, 0)\,\mathrm{e}^{-\mathrm{i}(ET+P'X)}. \tag{4.19}$$

The state $\Psi(0, 0)$ is a linear combination of multi-particle states $|p_1, p_2, \ldots, p_n\rangle$, where for each momentum p_i we need to specify the particle type too.

Such a phase dependence on the location of the centre of mass is standard in the quantum mechanics of one or more particles [19], but in field theory it is generally neglected. It is implicit in field theory, as one always regards $\mathrm{e}^{-\mathrm{i}Pa}$ as the operator that spatially displaces a state by a. With our notation we have, explicitly, $\mathrm{e}^{-\mathrm{i}Pa}\Psi(T, X) = \Psi(T, X+a)$.[2]

We can regard the pair of equations (4.17) and (4.18) as the Schrödinger equations of quantum field theory in 1+1 dimensions. In our supersymmetric field theory, the Hilbert space of states decomposes as $\mathcal{H} = \mathcal{H}_\mathrm{b} \oplus \mathcal{H}_\mathrm{f}$, where states in $\mathcal{H}_\mathrm{b}$ have even fermion number and states in $\mathcal{H}_\mathrm{f}$ have odd fermion number. Ψ can be a general element of $\mathcal{H}$, but normally one imposes the superselection rule that Ψ is either in $\mathcal{H}_\mathrm{b}$ or in $\mathcal{H}_\mathrm{f}$. The action of Q_1 and Q_2 maps states in $\mathcal{H}_\mathrm{b}$ to physically distinct states in $\mathcal{H}_\mathrm{f}$ and vice versa, degenerate in both energy and momentum.

This completes our summary of the standard interpretation of the field theory.

Now we show that, because of the supersymmetry, the Schrödinger equations of the field theory can be replaced by superevolution equations. To do this, we extend (1+1)-dimensional space–time to a superspace $\mathbb{R}^{2|2}$ with coordinates T, X, θ_1, θ_2, where θ_1 and θ_2 are odd. We introduce a state in superspace $\Psi(T, X, \theta_1, \theta_2)$ and consider its expansion in θ_1 and θ_2,

$$\Psi(T, X, \theta_1, \theta_2) = \Psi_0(T, X) + \theta_1\Psi_1(T, X) + \theta_2\Psi_2(T, X) + \theta_1\theta_2\Psi_{12}(T, X). \tag{4.20}$$

By analogy with what we did in supersymmetric quantum mechanics, we require that Ψ_0 and Ψ_{12} lie in $\mathcal{H}_\mathrm{b}$, and Ψ_1 and Ψ_2 lie in $\mathcal{H}_\mathrm{f}$. We also treat Ψ_1 and Ψ_2 as odd, anticommuting with θ_1 and θ_2.

[2] Another way in which the dependence is implicit in field theory is that the relationship between the particle creation operator $a_p^\dagger$ and the field operators $\phi(x)$ and $\pi(x)$ involves $\mathrm{e}^{\mathrm{i}px}$, and this changes phase if one displaces the spatial origin. So a one-particle state of momentum p changes under a displacement.

Our assumptions mean that the expansion of Ψ is analogous to that of a superfield in $\mathbb{R}^{2|2}$ whose components are classical bosonic and fermionic fields; however, here the components of Ψ are multi-particle quantum states.

The superspace evolution operators are

$$D_1 = \frac{\partial}{\partial\theta_1} - \theta_1 \mathrm{i}\partial_+ \tag{4.21}$$

and

$$D_2 = \frac{\partial}{\partial\theta_2} - \theta_2 \mathrm{i}\partial_-, \tag{4.22}$$

where $\partial_\pm = \partial_T \pm \partial_X$. They obey the relations

$$D_1^2 = -\mathrm{i}\partial_+, \tag{4.23}$$

$$D_2^2 = -\mathrm{i}\partial_-, \tag{4.24}$$

$$D_1 D_2 + D_2 D_1 = 0. \tag{4.25}$$

Because of the close formal similarity of this algebra with the supersymmetry algebra (4.10)–(4.12), we can impose the consistent pair of superevolution equations

$$D_1\Psi = Q_1\Psi, \tag{4.26}$$

$$D_2\Psi = Q_2\Psi. \tag{4.27}$$

By acting with D_1 and D_2 on each of these equations and noting that D_1 and D_2 anticommute with Q_1 and Q_2, we verify that these superevolution equations imply the Schrödinger equations (equivalent to (4.17) and (4.18))

$$\mathrm{i}\partial_+\Psi = (H+P)\Psi, \tag{4.28}$$

$$\mathrm{i}\partial_-\Psi = (H-P)\Psi. \tag{4.29}$$

It is worthwhile to expand both equations (4.26) and (4.27) in their components, to check their consistency. The first equation gives

$$\Psi_1 = Q_1\Psi_0, \tag{4.30}$$

$$\mathrm{i}\partial_+\Psi_0 = Q_1\Psi_1, \tag{4.31}$$

$$\Psi_{12} = -Q_1\Psi_2, \tag{4.32}$$

$$\mathrm{i}\partial_+\Psi_2 = -Q_1\Psi_{12}. \tag{4.33}$$

Since $Q_1^2 = H+P$, we can verify that each component state Ψ_0, Ψ_1, Ψ_2 and Ψ_{12} obeys (4.28). Similarly, the second superevolution equation gives

$$\Psi_2 = Q_2\Psi_0, \tag{4.34}$$

$$\mathrm{i}\partial_-\Psi_0 = Q_2\Psi_2, \tag{4.35}$$

$$\Psi_{12} = Q_2\Psi_1, \tag{4.36}$$

$$\mathrm{i}\partial_-\Psi_1 = Q_2\Psi_{12}, \tag{4.37}$$

which implies that each component obeys (4.29). From both sets of equations together, we see that the components are related algebraically to Ψ_0 by

$$(\Psi_0, \Psi_1, \Psi_2, \Psi_{12}) = (\Psi_0, Q_1\Psi_0, Q_2\Psi_0, -Q_1Q_2\Psi_0). \tag{4.38}$$

Provided Ψ_1, Ψ_2 and Ψ_{12} are related to Ψ_0 in this way, and Ψ_0 obeys the pair of Schrödinger equations (4.28) and (4.29), it follows that all the component equations above are satisfied.

So, as in supersymmetric quantum mechanics, the only independent state is Ψ_0, which is in the bosonic part of the Hilbert space, $\mathcal{H}_b$. Ψ_1, Ψ_2 and Ψ_{12} are shadow states that accompany Ψ_0 in the superevolution, but they carry no independent physical information. There is no physically independent fermionic state obtained by the action of Q_1 or Q_2 on a bosonic state.

We shall explore below, in a little more detail, the superevolution of particle states in free field theory.

5. Free field theory

The free theory of one scalar and one Majorana field, both of mass m, is diagonalized by passing to momentum space. One can directly see the particle content and can clarify the physics of the superevolution equations.

The scalar field operators $\pi(x)$ and $\phi(x)$ have the coupled momentum space expansions

$$\pi(x) = \int \frac{\mathrm{d}p}{2\pi}(-\mathrm{i})\sqrt{\frac{E_p}{2}}\left(c_p\,\mathrm{e}^{-\mathrm{i}px} - c_p^\dagger\,\mathrm{e}^{\mathrm{i}px}\right), \tag{5.1}$$

$$\phi(x) = \int \frac{\mathrm{d}p}{2\pi}\frac{1}{\sqrt{2E_p}}\left(c_p\,\mathrm{e}^{-\mathrm{i}px} + c_p^\dagger\,\mathrm{e}^{\mathrm{i}px}\right), \tag{5.2}$$

where $E_p = \sqrt{p^2+m^2}$.[3] The canonical commutation relations require

$$\left[c_p, c_{p'}^\dagger\right] = 2\pi\delta(p-p'), \tag{5.3}$$

$$[c_p, c_{p'}] = \left[c_p^\dagger, c_{p'}^\dagger\right] = 0. \tag{5.4}$$

For the Majorana field $\psi_\alpha(x)$ we first need to present the solutions of the classical Dirac equation

$$(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi = 0. \tag{5.5}$$

In the Majorana representation, this becomes

$$\partial_+\psi_2 = -m\psi_1, \tag{5.6}$$

$$\partial_-\psi_1 = m\psi_2. \tag{5.7}$$

Plane-wave solutions of positive frequency (energy) are of the form

$$\psi(t,x) = u(p)\,\mathrm{e}^{-\mathrm{i}(E_pt+px)}, \tag{5.8}$$

where

$$u(p) = \begin{pmatrix}\sqrt{E_p+p}\\ -\mathrm{i}\sqrt{E_p-p}\end{pmatrix}. \tag{5.9}$$

Similarly, there are negative-frequency plane-wave solutions

$$\psi(t,x) = v(p)\,\mathrm{e}^{\mathrm{i}(E_pt+px)}, \tag{5.10}$$

with

$$v(p) = \begin{pmatrix}\sqrt{E_p+p}\\ \mathrm{i}\sqrt{E_p-p}\end{pmatrix}. \tag{5.11}$$

The momentum space expansion of the Majorana field $\psi = \binom{\psi_1}{\psi_2}$ is

$$\psi(x) = \int \frac{\mathrm{d}p}{2\pi}\frac{1}{\sqrt{2E_p}}\left(a_pu(p)\,\mathrm{e}^{-\mathrm{i}px} + a_p^\dagger v(p)\,\mathrm{e}^{\mathrm{i}px}\right). \tag{5.12}$$

The expressions for $u(p)$ and $v(p)$ imply that both components of ψ are Hermitian. To satisfy the canonical anticommutation relations, one requires

$$\left\{a_p, a_{p'}^\dagger\right\} = 2\pi\delta(p-p'), \tag{5.13}$$

$$\{a_p, a_{p'}\} = \left\{a_p^\dagger, a_{p'}^\dagger\right\} = 0. \tag{5.14}$$

One also requires that the operators $c, c^\dagger$ commute with $a, a^\dagger$.

[3] (E_p, p) are the components of a covariant 2-vector p_μ.

Starting from formulae (4.6)–(4.9), with $W(\phi) = m\phi$, and performing a number of integrations, one can obtain the momentum space expressions for H, P, Q_1 and Q_2. These can be normal ordered without discarding infinite constants, because of the supersymmetry, and one finds

$$H \pm P = \int \frac{\mathrm{d}p}{2\pi}(E_p \pm p)\big(c_p^\dagger c_p + a_p^\dagger a_p\big), \tag{5.15}$$

$$Q_1 = \mathrm{i}\int \frac{\mathrm{d}p}{2\pi}\sqrt{E_p + p}\big(-a_p^\dagger c_p + c_p^\dagger a_p\big), \tag{5.16}$$

$$Q_2 = \int \frac{\mathrm{d}p}{2\pi}\sqrt{E_p - p}\big(a_p^\dagger c_p + c_p^\dagger a_p\big). \tag{5.17}$$

The (Fock) vacuum $|0\rangle$ is annihilated by c_p and a_p for all p. It is therefore annihilated by all the operators Q_1, Q_2, H and P, so it is supersymmetric and has zero energy and momentum.

Let us define bosonic and fermionic one-particle states by

$$|p_\mathrm{b}\rangle = c_p^\dagger|0\rangle, \qquad |p_\mathrm{f}\rangle = a_p^\dagger|0\rangle. \tag{5.18}$$

As a phase convention, we say these states have their centre of mass at the origin, and we ignore the additional factors of $\sqrt{2E_p}$ required for a relativistic normalization. There is just one solution of the superevolution equations that can be constructed from these. Ψ_0 should be bosonic, so, at $(T, X) = (0, 0)$, we set it equal to $|p_\mathrm{b}\rangle$. Then, by acting with Q_1 and Q_2, as given by (5.16) and (5.17), we find that

$$\Psi_1 = -\mathrm{i}\sqrt{E_p + p}|p_\mathrm{f}\rangle, \qquad \Psi_2 = \sqrt{E_p - p}|p_\mathrm{f}\rangle, \qquad \Psi_{12} = -\mathrm{i}m|p_\mathrm{b}\rangle. \tag{5.19}$$

We must multiply all these component states by $\mathrm{e}^{-(E_pT+pX)}$ to obtain their values at (T, X). Superevolution implies that the only physical particle is a scalar boson, although its shadow states Ψ_1 and Ψ_2 are fermionic.

Two-particle states can be constructed similarly. The obvious two-boson state is $\Psi_0 = c_p^\dagger c_{p'}^\dagger|0\rangle$, with energy $E = E_p + E_{p'}$ and momentum $P' = p + p'$. This generates a solution of the superevolution equations with the shadow states

$$\Psi_1 = -\mathrm{i}\sqrt{E_p + p}a_p^\dagger c_{p'}^\dagger|0\rangle - \mathrm{i}\sqrt{E_{p'} + p'}a_{p'}^\dagger c_p^\dagger|0\rangle, \tag{5.20}$$

$$\Psi_2 = \sqrt{E_p - p}a_p^\dagger c_{p'}^\dagger|0\rangle + \sqrt{E_{p'} - p'}a_{p'}^\dagger c_p^\dagger|0\rangle, \tag{5.21}$$

$$\Psi_{12} = -\mathrm{i}\big(\sqrt{(E_{p'} + p')(E_p - p)} - \sqrt{(E_p + p)(E_{p'} - p')}\big)a_p^\dagger a_{p'}^\dagger|0\rangle - 2\mathrm{i}mc_p^\dagger c_{p'}^\dagger|0\rangle. \tag{5.22}$$

All these must be multiplied by $\mathrm{e}^{-(ET+P'X)}$. The coefficient of the first term in Ψ_{12} simplifies to $\mathrm{i}\sqrt{2(E_pE_{p'} - pp' - m^2)}$, where the positive square root is taken if $p > p'$ and the negative root if $p < p'$. A further simplification is possible by introducing a rapidity variable λ, such that $E_p = m\cosh\lambda$ and $p = m\sinh\lambda$, and similarly λ'. Then this coefficient becomes $2\mathrm{i}m\sinh\frac{1}{2}(\lambda - \lambda')$. Note that not only Ψ_0, but also all the shadow states are symmetric under the interchange of p and p'.

There is a further candidate state for Ψ_0, namely $\Psi_0 = a_p^\dagger a_{p'}^\dagger|0\rangle$, which is also in $\mathcal{H}_\mathrm{b}$, since it is a two-fermion state. The shadow states associated with this are rather similar to those given above. This state, and similar multi-particle states with an even number of fermions, are the most problematic for our superevolution proposal. We were hoping for an interpretation of supersymmetric field theory with only one type of particle. We have managed to exclude the one-fermion state, but need to allow two-fermion states.

We have the following thoughts about this problem. First, note that the state $a_p^\dagger a_{p'}^\dagger|0\rangle$ is not directly related to $c_p^\dagger c_{p'}^\dagger|0\rangle$ by supersymmetry (although it occurs in combination with

$c_p^\dagger c_{p'}^\dagger|0\rangle$ in Ψ_{12} above), and it is probably an accident of the free field theory that these two states are degenerate in energy and momentum. In the interacting theory, the two-boson sector and the two-fermion sector may be physically quite different, having different two-particle to two-particle scattering amplitudes, and different bound states (if any). This would follow from the different permutational symmetry. Both in the free and interacting theories, the two-boson states are symmetric under particle interchange and the two-fermion states are antisymmetric. As a shortcut to ensure that there is only one type of physical particle, we could perhaps require that all multi-particle states are totally symmetric. This proposal is consistent with the superevolution equations, because the action of Q_1 and Q_2 preserves the symmetry of states, but whether it is consistent in the interacting theory requires further investigation.

6. Conclusion

We have revived the idea that the fundamental evolution equation in supersymmetric quantum mechanics should be a 'square root' of the Schrödinger equation. This means treating the supersymmetry charge as an evolution operator in a superspace, and we call the resulting equations the superevolution equations. The supersymmetry algebra implies that if the superevolution equations are satisfied then so is the Schrödinger equation. Usually, in supersymmetric quantum mechanics, there are degenerate bosonic and fermionic states which are physically distinct and linearly independent, but the superevolution equations relate them, so the degeneracy disappears. One version of the superevolution takes place in a rather abstract superspace, but in Witten's model of supersymmetric quantum mechanics, the superevolution equations can be presented using standard techniques from the theory of differential forms.

We have extended the notion of superevolution to a simple supersymmetric field theory in 1+1 dimensions. To make this work we needed to clarify the idea that the Schrödinger equation in quantum field theory determines the evolution of states in both time and space (via the Hamiltonian and total momentum operators). The superevolution equations use the supercharges to define a consistent superspace evolution. Again, the superevolution equations imply that the Schrödinger equation is satisfied, but the space of solutions is smaller, because the superevolution relates states that are usually treated as physically independent. As a result, there is a suppression of the degeneracy between bosonic and fermionic one-particle states. A natural choice leads to a unique supersymmetric vacuum of zero energy and momentum, and the only physical one-particle state being bosonic. Two-particle bosonic states also occur, as desired, but it could be problematic to remove the two-fermion states. We suggested a way to deal with these too, leading to a theory which retains its supersymmetric character, but which has only bosonic physical particles.

One might ask, in this case, what the fermions are doing. They would contribute internal lines to Feynman diagrams (the vertices are determined by the interaction term $2\mathrm{i}\frac{\mathrm{d}W(\phi)}{\mathrm{d}\phi}\psi_1\psi_2$ of the Hamiltonian). The best interpretation might be that the supersymmetric theory defines a special way of quantizing the purely bosonic field theory, leading to all the usual advantages of supersymmetry (finiteness, zero vacuum energy), but without physical fermions. The fermions are then rather like the ghosts that occur in gauge theories (but we prefer to call them shadows).

It is of course important to explore extensions of the ideas here to higher dimensions and to investigate whether it is possible to have a supersymmetric interpretation of a theory with just fermions, or of a theory like QED, which has spin-1 photons and spin-$\frac{1}{2}$ (non-Majorana) electrons.

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